TWO REGULARITY CRITERIA FOR THE 3D MHD EQUATIONS

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Abstract. This work establishes two regularity criteria for the 3D incompressible MHD equations. The first one is in terms of the derivative of the velocity field in one-direction while the second one requires suitable boundedness of the derivative of the pressure in one-direction.

1. Introduction

This paper is concerned with the global regularity of solutions to the 3D incompressible magnetohydrodynamical (MHD) equations

$$u_t + u\cdot\nabla u = \nu\,\Delta u - \nabla p + b\cdot\nabla b, \qquad x\in \mathbf{R}^3,\ t>0, \tag{1.1}$$

$$b_t + u\cdot\nabla b = \eta\,\Delta b + b\cdot\nabla u, \qquad x\in \mathbf{R}^3,\ t>0, \tag{1.2}$$

$$\nabla\cdot u = 0, \qquad x\in \mathbf{R}^3,\ t>0, \tag{1.3}$$

$$\nabla\cdot b = 0, \qquad x\in \mathbf{R}^3,\ t>0, \tag{1.4}$$

where u is the fluid velocity, b the magnetic field, p the pressure, ν the viscosity and η the magnetic diffusivity. Without loss of generality, we set $\nu=\eta=1$ in the rest of the paper. The MHD equations govern the dynamics of the velocity and magnetic fields in electrically conducting fluids such as plasmas. (1.1) reflects the conservation of momentum, (1.2) is the induction equation and (1.3) specifies the conservation of mass. Besides their physical applications, the MHD equations are also mathematically significant. Fundamental mathematical issues such as the global regularity of their solutions have generated extensive research and many interesting results have been obtained (see, e.g., [2],[7],[11],[13],[16],[17], [18],[19],[23],[26],[27],[28],[34], [35],[36],[38],[39],[40],[43],[45], [47],[48],[49],[50],[51],[52],[55]).

Attention here is focused on the global regularity of solutions to the initial-value problem (IVP) of (1.1),(1.2),(1.3) and (1.4) with a given initial data

$$u(x,0)=u_0(x), \quad b(x,0)=b_0(x), \qquad x\in\mathbf{R}^3. \tag{1.5}$$

It is currently unknown whether solutions of this IVP can develop finite time singularities even if (u_0,b_0) is sufficiently smooth. This work presents new regularity criteria under which the regularity of the solution is preserved for all time. The global regularity issue has been thoroughly investigated for the 3D Navier-Stokes equations and many important regularity criteria have been established (see, e.g., [3],[4],[5],[6],[8],[9],[10],[12], [14],[15],[20],[21],[25],[29],[30], [31],[32],[37],[41],[42],[44],[46],[53],[54]). Some of these criteria can be extended to the 3D MHD equations by making assumptions on both u and b (see, e.g., [7],[47]). Realizing the dominant role played by the velocity field in the regularity issue, He and Xin was able to derive criteria in terms of the velocity field u alone ([27],[28]). They showed that, if u satisfies

$$\int_0^T \|\nabla u(\cdot,t)\|_\alpha^\beta\,dt < \infty \quad \text{with} \quad \frac{3}{\alpha}+\frac{2}{\beta}=2 \quad\text{and}\quad 1<\beta\le 2, \tag{1.6}$$

then the solution (u,b) is regular on $[0,T]$. This assumption was weakened in [51] with L^α-norm replaced by norms in Besov spaces and further improved by Chen, Miao and Zhang in [17]. As pointed out in [27], the regularity criteria in terms of the velocity field alone are consistent with the numerical simulations in [40] and with the observations of space and laboratory plasmas in [24].

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This paper presents two new regularity criteria. The first one assumes

$$\int_0^T \|u_z(\cdot,t)\|_\alpha^\beta \, dt < \infty \quad \text{with} \quad \alpha \ge 3 \quad \text{and} \quad \frac{3}{\alpha} + \frac{2}{\beta} \le 1 \tag{1.7}$$

and the second requires the pressure satisfy

$$\int_0^T \|p_z(\tau)\|_\alpha^\beta \, d\tau < \infty \quad \text{with} \quad \alpha \ge \frac{12}{7} \quad \text{and} \quad \frac{3}{\alpha} + \frac{2}{\beta} \le \frac{7}{4}. \tag{1.8}$$

That is, any solution (u, b) of the 3D MHD equations is regular if the derivative of u in one direction, say along the z-axis, is bounded in $L^\beta([0,T]; L^\alpha)$ with (α, β) satisfying (1.7) or if the derivative of p in one direction satisfies (1.8). The proof of the first criterion is accomplished through two stages with the first controlling the time integrals of $\|\nabla u_z\|_2$ and $\|\nabla b_z\|$ in terms of the $L^\beta([0,T]; L^\alpha)$-norm of u_z and the second bounding $\|\nabla u\|_2$ and $\|\nabla b\|_2$ by the time integrals of $\|\nabla u_z\|_2$ and $\|\nabla b_z\|$. The details are presented in the second section. The criterion in terms of p_z and its proof are provided in the third section.

We will use the following elementary inequalities:

$$\|\phi\|_\gamma \le C \, \|\phi_x\|_\lambda^{\frac{1}{3}} \, \|\phi_y\|_\lambda^{\frac{1}{3}} \, \|\phi_z\|_\mu^{\frac{1}{3}}, \tag{1.9}$$

where the parameters μ, λ and γ satisfy

$$1 \le \mu, \ \lambda < \infty, \quad 1 < \frac{1}{\mu} + \frac{2}{\lambda} \le 4 \quad \text{and} \quad \gamma = \frac{3\lambda}{2 - \lambda\left(1 - \frac{1}{\mu}\right)},$$

and

$$\|\phi\|_r \le C(r) \, \|\phi\|_2^{\frac{6-r}{2r}} \, \|\phi_x\|_2^{\frac{r-2}{2r}} \, \|\phi_y\|_2^{\frac{r-2}{2r}} \, \|\phi_z\|_2^{\frac{r-2}{2r}}, \quad 2 \le r \le 6. \tag{1.10}$$

These inequalities may be found in [1],[22],[33]. For the convenience of the readers, the proofs of these inequalities are provided in Appendix A. Throughout the rest of this paper the L^p-norm of a function f is denoted by $\|f\|_p$, the H^s-norm by $\|f\|_{H^s}$ and the norm in the Sobolev space $W^{s,p}$ by $\|f\|_{W^{s,p}}$.

2. Criterion in terms of u_z

This section establishes the regularity criteria in terms of u_z.

Theorem 2.1. *Assume $(u_0, b_0) \in H^3$, $\nabla \cdot u_0 = 0$ and $\nabla \cdot b_0 = 0$. Let (u, b) be the corresponding solution of the 3D MHD equations (1.1),(1.2),(1.3) and (1.4). If u satisfies*

$$M(T) \equiv \int_0^T \|u_z(\cdot,t)\|_\alpha^\beta \, dt < \infty \quad \textit{with} \quad \alpha \ge 3 \quad \textit{and} \quad \frac{3}{\alpha} + \frac{2}{\beta} \le 1 \tag{2.1}$$

for some $T > 0$, then (u, b) can be extended to the time interval $[0, T+\epsilon)$ for some $\epsilon > 0$.

The proof of this theorem is divided into two major parts. The first part establishes bounds for $\|u_z\|_2$, $\|b_z\|_2$ and the time integrals of $\|\nabla u_z\|_2^2$ and $\|\nabla b_z\|_2^2$ while the second controls $\|\nabla u\|_2$ and $\|\nabla b\|_2$ in terms of the time integrals of $\|\nabla u_z\|_2^2$ and $\|\nabla b_z\|_2^2$.

2.1. Bounds for $\|u_z\|_2$ and $\|b_z\|_2$.

This subsection bounds $\|u_z\|_2$ and $\|b_z\|_2$ in terms of M in (2.1).

Proposition 2.2. *Assume $(u_0, b_0) \in H^3$, $\nabla \cdot u_0 = 0$ and $\nabla \cdot b_0 = 0$. Let (u, b) be the corresponding solution of the 3D MHD equations (1.1),(1.2),(1.3) and (1.4). Suppose (2.1) holds. Then, for any $t \le T$,*

$$\begin{aligned} \|u_z(t)\|_2^2 + \|b_z(t)\|_2^2 &\le C \, e^{(\|u_0\|_2^2 + \|b_0\|_2^2)} \, e^{M(t)} \\ &\quad \times \left[\left(\|u_z(0)\|_2^2 + \|b_z(0)\|_2^2\right)^{\frac{3}{2\alpha-3}} + C\left(\|u_0\|_2^2 + \|b_0\|_2^2 + M(t)\right) \right]^{\frac{2\alpha-3}{3}} \end{aligned} \tag{2.2}$$

and

$$\int_0^t \left(\|\nabla u_z(\tau)\|_2^2 + \|\nabla b_z(\tau)\|_2^2 \right) d\tau \le F(M(t)) < \infty, \tag{2.3}$$

where $F(M(t))$ is an explicit function of $M(t)$.

Proof. It is easy to see that (u,b) satisfies

$$\|u(t)\|_2^2 + \|b(t)\|_2^2 + 2\int_0^t (\|\nabla u(\tau)\|_2^2 + \|\nabla b(\tau)\|_2^2)\, d\tau \le \|u_0\|_2^2 + \|b_0\|_2^2. \tag{2.4}$$

Adding the inner products of u_z with ∂_z of (1.1) and of b_z with ∂_z of (1.2), we obtain, after integration by parts,

$$\begin{aligned}
&\frac{1}{2}\,\frac{d\,(\|u_z\|_2^2 + \|b_z\|_2^2)}{dt} + \|\nabla u_z\|_2^2 + \|\nabla b_z\|_2^2 \\
&\quad = -\int [(u_z\cdot\nabla u)\cdot u_z - (b_z\cdot\nabla b)\cdot u_z + (u_z\cdot\nabla b)\cdot b_z - (b_z\cdot\nabla u)\cdot b_z]\, dxdydz \\
&\quad \equiv I_1 + I_2 + I_3 + I_4.
\end{aligned}$$

To bound I_1, we integrate by parts and apply Hölder's inequality to obtain

$$|I_1| = \left|\int (u_z\cdot\nabla u_z)\cdot u\right| \le C\|\nabla u_z\|_2\,\|u_z\|_r\,\|u\|_{3\alpha},$$

where we have omitted $dxdydz$ in the integral for notational convenience and r satisfies

$$2\le r\le 6, \quad \frac{1}{r} + \frac{1}{3\alpha} = \frac{1}{2}. \tag{2.5}$$

Applying the Sobolev inequality

$$\|u_z\|_r \le C\,\|u_z\|_2^{1-3(\frac12-\frac1r)}\,\|\nabla u_z\|_2^{3(\frac12-\frac1r)}$$

and bounding $\|u\|_{3\alpha}$ by (1.9), we find

$$|I_1| \le C\,\|\nabla u_z\|_2^{1+3(\frac12-\frac1r)}\,\|u_z\|_2^{1-3(\frac12-\frac1r)}\,\|u_z\|_\alpha^{\frac13}\,\|\nabla u\|_2^{\frac23}.$$

By Young's inequality,

$$|I_1| \quad\le\quad \frac14\,\|\nabla u_z\|_2^2 + C\,\|u_z\|_2^2\,\|u_z\|_\alpha^q\,\|\nabla u\|_2^{2q}$$

with

$$q = \frac{2}{3-9(\frac12-\frac1r)} = \frac{2}{3(1-\frac1\alpha)}. \tag{2.6}$$

When $\alpha\ge 3$, we have $2q\le 2$ and another application of Young's inequality implies

$$|I_1| \le \frac14\,\|\nabla u_z\|_2^2 + C\,\|u_z\|_2^2\,\left(\|u_z\|_\alpha^\gamma + \|\nabla u\|_2^2\right),$$

where

$$\gamma \equiv \frac{q}{1-q} = \frac{2}{1-\frac{3}{\alpha}} \quad\text{or}\quad \frac{3}{\alpha}+\frac{2}{\gamma} = 1.$$

We now bound I_2. By Hölder's, Sobolev's and Young's inequalities,

$$\begin{aligned}
|I_2| &\le C\,\|\nabla b\|_2\,\|u_z\|_\alpha\,\|b_z\|_{\frac{2\alpha}{\alpha-2}} \\
&\le C\,\|\nabla b\|_2\,\|u_z\|_\alpha\,\|b_z\|_2^{1-\frac{3}{\alpha}}\,\|\nabla b_z\|_2^{\frac{3}{\alpha}} \\
&\le \frac14\,\|\nabla b_z\|_2^2 + C\,\|\nabla b\|_2^{\frac{2\alpha}{2\alpha-3}}\,\|u_z\|_\alpha^{\frac{2\alpha}{2\alpha-3}}\,\|b_z\|_2^{\frac{2\alpha-6}{2\alpha-3}} \\
&\le \frac14\,\|\nabla b_z\|_2^2 + C\,\left(\|\nabla b\|_2^2 + \|u_z\|_\alpha^\gamma\right)\,\|b_z\|_2^{\frac{2\alpha-6}{2\alpha-3}}
\end{aligned}$$

where

$$\gamma = \frac{2\alpha}{\alpha-3} \quad\text{or}\quad \frac{3}{\alpha}+\frac{2}{\gamma} = 1.$$

I_3 can be bounded exactly as I_2. To bound I_4, we integrate by parts and apply Hölder's inequality,

$$I_4 = -\int [(b_z\cdot\nabla u)\cdot b_z] = \int [(b_z\cdot\nabla b_z)\cdot u] \le \|\nabla b_z\|_2\,\|b_z\|_r\,\|u\|_{3\alpha},$$

where $\frac{1}{r}+\frac{1}{3\alpha}=\frac{1}{2}$. Following the steps as in the bound of I_1, we have

$$|I_4| \le \frac{1}{4}\,\|\nabla b_z\|_2^2 + C\,\left(\|u_z\|_\alpha^\gamma + \|\nabla u\|_2^2\right)\,\|b_z\|_2^2.$$

Combining the estimates for I_1, I_2, I_3 and I_4, we find

$$\begin{aligned}&\frac{d\left(\|u_z\|_2^2+\|b_z\|_2^2\right)}{dt} + \|\nabla u_z\|_2^2 + \|\nabla b_z\|_2^2 \\ &\quad\le C\left(\|u_z\|_\alpha^\gamma + \|\nabla u\|_2^2\right)\,\left(\|u_z\|_2^2+\|b_z\|_2^2\right) + C\left(\|\nabla b\|_2^2 + \|u_z\|_\alpha^\gamma\right)\,\|b_z\|_2^{\frac{2\alpha-6}{2\alpha-3}}. \end{aligned}\tag{2.7}$$

(2.2) and (2.3) then follows from (2.4), (2.7) and Gronwall's inequality. □

2.2. **Bounds for $\|\nabla u\|_2$ and $\|\nabla b\|_2$.** This subsection establishes bounds for $\|\nabla u\|_2$ and $\|\nabla b\|_2$.

Proposition 2.3. *Assume $(u_0,b_0)\in H^3$, $\nabla\cdot u_0=0$ and $\nabla\cdot b_0=0$. Let (u,b) be the corresponding solution of the 3D MHD equations (1.1),(1.2),(1.3) and (1.4). Suppose (2.1) holds. Then, for any $t\le T$,*

$$\|\nabla u(t)\|_2^2 + \|\nabla b(t)\|_2^2 + \int_0^t \left(\|\Delta u(\tau)\|_2^2 + \|\Delta b(\tau)\|_2^2\right) d\tau \le G(M(t)) < \infty,$$

where $G(M(t))$ denotes an explicit function of $M(t)$.

Proof. Adding the inner products of (1.1) with Δu and of (1.2) with Δb and integrating by parts, we have

$$\frac{1}{2}\,\frac{d}{dt}\left(\|\nabla u\|_2^2 + \|\nabla b\|_2^2\right) + \|\Delta u\|_2^2 + \|\Delta b\|_2^2 \tag{2.8}$$

$$= -\int u\cdot\nabla u\cdot\Delta u + \int b\cdot\nabla b\cdot\Delta u - \int u\cdot\nabla b\cdot\Delta b + \int b\cdot\nabla u\cdot\Delta b. \tag{2.9}$$

By further integrating by parts, we obtain

$$-\int u\cdot\nabla u\cdot\Delta u + \int b\cdot\nabla b\cdot\Delta u - \int u\cdot\nabla b\cdot\Delta b + \int b\cdot\nabla u\cdot\Delta b \le \|\nabla u\|_3^3 + 3\|\nabla u\|_3\,\|\nabla b\|_3^2.$$

By (1.10),

$$\|\nabla u\|_3^3 \le C\,\left(\|\nabla u\|_2^{\frac{1}{2}}\,\|\nabla_h\nabla u\|_2^{\frac{1}{3}}\,\|\nabla u_z\|_2^{\frac{1}{6}}\right)^3,$$

where $\nabla_h \equiv (\partial_x,\partial_y)$. By Young's inequality,

$$\|\nabla u\|_3^3 \le \frac{1}{4}\|\nabla_h\nabla u\|_2^2 + C\,\|\nabla u\|_2^3\,\|\nabla u_z\|_2 \le \frac{1}{4}\|\nabla_h\nabla u\|_2^2 + C\,\left(\|\nabla u\|_2^2 + \|\nabla u_z\|_2^2\right)\,\|\nabla u\|_2^2.$$

Similarly,

$$\|\nabla u\|_3\,\|\nabla b\|_3^2 \le \frac{1}{4}\|\nabla_h\nabla u\|_2^2 + \frac{1}{2}\|\nabla_h\nabla b\|_2^2 + C\left(\|\nabla u\|_2^2 + \|\nabla u_z\|_2^2 + \|\nabla b_z\|_2^2\right)\|\nabla b\|_2^2.$$

Therefore,

$$\frac{d}{dt}\left(\|\nabla u\|_2^2 + \|\nabla b\|_2^2\right) + \|\Delta u\|_2^2 + \|\Delta b\|_2^2 \le C\,\left(\|\nabla u\|_2^2 + \|\nabla u_z\|_2^2 + \|\nabla b_z\|_2^2\right)\left(\|\nabla u\|_2^2 + \|\nabla b\|_2^2\right).$$

Gronwall's inequality coupled with Proposition 2.2 then yields the desired bounds. □

3. Criterion in terms of p_z

This section presents the regularity criterion with an assumption on p_z.

Theorem 3.1. *Assume the initial data $(u_0, b_0) \in H^1 \cap L^4$, $\nabla \cdot u_0 = 0$ and $\nabla \cdot b_0 = 0$. Let (u, b) be the corresponding solution of the 3D MHD equations (1.1),(1.2),(1.3) and (1.4). If the pressure p associated with the solution satisfies*

$$\int_0^T \|p_z(\tau)\|_\alpha^\beta \, d\tau < \infty \quad \text{with} \quad \alpha \ge \frac{12}{7} \quad \text{and} \quad \frac{3}{\alpha} + \frac{2}{\beta} \le \frac{7}{4} \tag{3.1}$$

for some $T > 0$, then (u, b) remains regular on $[0, T]$, namely $(u, b) \in C([0, T]; H^1 \cap L^4)$.

Since higher-order Sobolev norms of (u, b) can be controlled by its H^1-norm (see e.g. [45]), a special consequence of this theorem is that (3.1) yields the global regularity of classical solutions. To prove this theorem, we establish the L^4-bound of (u, b) and the desired regularity then follows from the standard Serrin type criteria on the 3D MHD equations [48].

Proposition 3.2. *Assume the initial data $(u_0, b_0) \in H^1 \cap L^4$, $\nabla \cdot u_0 = 0$ and $\nabla \cdot b_0 = 0$. Let (u, b) be the corresponding solution of the 3D MHD equations (1.1),(1.2),(1.3) and (1.4). If the pressure p satisfies (3.1), then (u, b) obeys the bound*

$$\begin{aligned} \|w^+\|_4^4 + \|w^-\|_4^4 \quad &+ \quad \int_0^t \left(\|\nabla |w^+|^2\|_2^2 + \|\nabla |w^-|^2\|_2^2\right) \, d\tau \\ &+ \quad 4 \int_0^t \int \left(|w^+|^2 \, |\nabla w^+|^2 + |w^-|^2 \, |\nabla w^-|^2\right) \, dxdydz \, d\tau < \infty \end{aligned}$$

for any $t \le T$, where

$$w^\pm = u \pm b.$$

Proof of Proposition 3.2. We first convert the MHD equations into a symmetric form. Adding and subtracting (1.1) and (1.2), we find that w^+ and w^- satisfy

$$\partial_t w^+ + w^- \cdot \nabla w^+ = \Delta w^+ - \nabla p, \tag{3.2}$$

$$\partial_t w^- + w^+ \cdot \nabla w^- = \Delta w^- - \nabla p, \tag{3.3}$$

$$\nabla \cdot w^+ = 0, \quad \nabla \cdot w^- = 0. \tag{3.4}$$

Adding the inner products of (3.2) with $w^+ \, |w^+|^2$ and of (3.3) with $w^- \, |w^-|^2$ and integrating by parts, we find

$$\begin{aligned} &\frac{1}{4} \frac{d}{dt} \left(\|w^+\|_4^4 + \|w^-\|_4^4\right) + \frac{1}{2} \left(\|\nabla |w^+|^2\|_2^2 + \|\nabla |w^-|^2\|_2^2\right) + \int \left(|w^+|^2 \, |\nabla w^+|^2 + |w^-|^2 \, |\nabla w^-|^2\right) \\ &= J_1 + J_2, \end{aligned} \tag{3.5}$$

where

$$J_1 = \int p \, w^+ \cdot \nabla |w^+|^2, \quad J_2 = \int p \, w^- \cdot \nabla |w^-|^2.$$

By Hölder's inequality,

$$J_1 \le C \, \|p\|_4 \, \|w^+\|_4 \, \|\nabla |w^+|^2\|_2.$$

We choose λ such that

$$\frac{1}{\alpha} + \frac{2}{\lambda} = \frac{7}{4} \quad \text{or} \quad \frac{3\lambda}{2 - \lambda \left(1 - \frac{1}{\alpha}\right)} = 4.$$

It then follows from (1.9) that

$$\|p\|_4 \le C \, \|p_z\|_\alpha^{\frac{1}{3}} \, \|\nabla p\|_\lambda^{\frac{2}{3}}.$$

To further bound $\|\nabla p\|_\lambda$, we take the divergence of (3.2) to obtain

$$\Delta p = -\nabla \cdot (w^- \cdot \nabla w^+).$$

By Hölder's inequality,

$$\|\nabla p\|_\lambda \le C \, \|w^-\|_{\frac{2\lambda}{2-\lambda}} \, \|\nabla w^+\|_2.$$

Furthermore, by Sobolev 's inequality,

$$\|w^-\|_{\frac{2\lambda}{2-\lambda}} = \||w^-|^2\|^{\frac{1}{2}}_{\frac{\lambda}{2-\lambda}} \le C\, \||w^-|^2\|_2^{\frac{3}{\lambda}-\frac{7}{4}}\, \|\nabla|w^-|^2\|_2^{\frac{9}{4}-\frac{3}{\lambda}} = C\, \|w^-\|_4^{\frac{6}{\lambda}-\frac{7}{2}}\, \|\nabla|w^-|^2\|_2^{\frac{9}{4}-\frac{3}{\lambda}},$$

where we have used the fact that $\frac{12}{7} \le \alpha$ and thus $\lambda \le \frac{12}{7}$. Therefore,

$$\|p\|_4 \le C\, \|p_z\|_\alpha^{\frac{1}{3}}\, \|\nabla w^+\|_2^{\frac{2}{3}}\, \|w^-\|_4^{\frac{4}{\lambda}-\frac{7}{3}}\, \|\nabla|w^-|^2\|_2^{\frac{3}{2}-\frac{2}{\lambda}}$$

and thus

$$J_1 \le C\, \|p_z\|_\alpha^{\frac{1}{3}}\, \|\nabla w^+\|_2^{\frac{2}{3}}\, \|w^-\|_4^{\frac{4}{\lambda}-\frac{7}{3}}\, \|\nabla|w^-|^2\|_2^{\frac{3}{2}-\frac{2}{\lambda}}\, \|w^+\|_4\, \|\nabla|w^+|^2\|_2.$$

By Young's inequality,

$$\begin{aligned} J_1 \quad &\le \quad \frac{1}{8}\,\|\nabla|w^+|^2\|_2^2 + \frac{1}{8}\,\|\nabla|w^-|^2\|_2^2 \\ &\quad + C\, \|p_z\|_\alpha^{\frac{4\lambda}{3(4-\lambda)}}\, \|\nabla w^+\|_2^{\frac{8\lambda}{3(4-\lambda)}}\, \|w^-\|_4^{\frac{4(12-7\lambda)}{3(4-\lambda)}}\, \|w^+\|_4^{\frac{4\lambda}{4-\lambda}}. \end{aligned}$$

Further applications of Young's inequality imply

$$\begin{aligned} &\|p_z\|_\alpha^{\frac{4\lambda}{3(4-\lambda)}}\, \|\nabla w^+\|_2^{\frac{8\lambda}{3(4-\lambda)}} \le \|p_z\|_\alpha^{\frac{4\lambda}{12-7\lambda}} + \|\nabla w^+\|_2^2, \\ &\|w^-\|_4^{\frac{4(12-7\lambda)}{3(4-\lambda)}}\, \|w^+\|_4^{\frac{4\lambda}{4-\lambda}} \le \|w^+\|_4^4 + \|w^-\|_4^{\frac{2(12-7\lambda)}{3(2-\lambda)}}. \end{aligned}$$

Since $\frac{2(12-7\lambda)}{3(2-\lambda)} < 4$, we obtain without loss of generality that

$$\begin{aligned} J_1 \quad &\le \quad \frac{1}{8}\,\|\nabla|w^+|^2\|_2^2 + \frac{1}{8}\,\|\nabla|w^-|^2\|_2^2 \\ &\quad + C\, \left(\|p_z\|_\alpha^{\frac{4\lambda}{12-7\lambda}} + \|\nabla w^+\|_2^2\right) \left(\|w^+\|_4^4 + \|w^-\|_4^4\right). \end{aligned} \tag{3.6}$$

Similarly, we have

$$\begin{aligned} J_2 \quad &= \quad \int p\, w^- \cdot \nabla|w^-|^2 \\ &\le \quad \frac{1}{8}\,\|\nabla|w^+|^2\|_2^2 + \frac{1}{8}\,\|\nabla|w^-|^2\|_2^2 \\ &\quad + C\, \left(\|p_z\|_\alpha^{\frac{4\lambda}{12-7\lambda}} + \|\nabla w^-\|_2^2\right) \left(\|w^+\|_4^4 + \|w^-\|_4^4\right). \end{aligned} \tag{3.7}$$

Inserting (3.6) and (3.7) in (3.5) and applying Gronwall's inequality, we obtain the desired result. □

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Appendix A.

This appendix provides the proofs of the inequalities (1.9) and (1.10). For the convenience of future references, we write these inequalities as lemmas.

Lemma A.1. *Let μ, λ and γ be three parameters that satisfy*

$$1 \le \mu,\ \lambda < \infty, \quad 1 < \frac{1}{\mu} + \frac{2}{\lambda} \le 4 \quad \textit{and} \quad \gamma = \frac{3\lambda}{2 - \lambda\left(1 - \frac{1}{\mu}\right)}.$$

Assume $\phi \in H^1(\mathbf{R}^3)$, $\phi_x,\ \phi_y \in L^\lambda(\mathbf{R}^3)$ and $\phi_z \in L^\mu(\mathbf{R}^3)$. Then, there exists a constant $C = C(\mu, \lambda)$ such that

$$\|\phi\|_\gamma \le C\ \|\phi_x\|_\lambda^{\frac{1}{3}}\ \|\phi_y\|_\lambda^{\frac{1}{3}}\ \|\phi_z\|_\mu^{\frac{1}{3}}. \tag{A.1}$$

Especially, when $\lambda = 2$, there exists a constant $C = C(\mu)$ such that

$$\|\phi\|_{3\mu} \le C\ \|\phi_x\|_2^{\frac{1}{3}}\ \|\phi_y\|_2^{\frac{1}{3}}\ \|\phi_z\|_\mu^{\frac{1}{3}}, \tag{A.2}$$

which holds for any $\phi \in H^1(\mathbf{R}^3)$ and $\phi_z \in L^\mu(\mathbf{R}^3)$ with $1 \le \mu < \infty$.

Proof. Clearly,

$$|\phi(x,y,z)|^{1+(1-\frac{1}{\lambda})\gamma} \le C\int_{-\infty}^{x} |\phi(t,y,z)|^{(1-\frac{1}{\lambda})\gamma}|\partial_t\phi(t,y,z)|\ dt, \tag{A.3}$$

$$|\phi(x,y,z)|^{1+(1-\frac{1}{\lambda})\gamma} \le C\int_{-\infty}^{y} |\phi(x,t,z)|^{(1-\frac{1}{\lambda})\gamma}|\partial_t\phi(x,t,z)|\ dt, \tag{A.4}$$

$$|\phi(x,y,z)|^{1+(1-\frac{1}{\mu})\gamma} \le C\int_{-\infty}^{z} |\phi(x,y,t)|^{(1-\frac{1}{\mu})\gamma}|\partial_t\phi(x,y,t)|\ dt. \tag{A.5}$$

Therefore,

$$\begin{aligned}|\phi(x,y,z)|^\gamma \le\ & C\left[\int_{-\infty}^{\infty} |\phi(x,y,z)|^{(1-\frac{1}{\lambda})\gamma}|\partial_x\phi(x,y,z)|\ dx\right]^{\frac{1}{2}}\left[\int_{-\infty}^{\infty} |\phi(x,y,z)|^{(1-\frac{1}{\lambda})\gamma}|\partial_y\phi(x,y,z)|\ dy\right]^{\frac{1}{2}}\\ &\times\left[\int_{-\infty}^{\infty} |\phi(x,y,z)|^{(1-\frac{1}{\mu})\gamma}|\partial_z\phi(x,y,z)|\ dz\right]^{\frac{1}{2}}.\end{aligned}$$

Integrating with respect to x and applying Hölder's inequality, we have

$$\begin{aligned}\int_{-\infty}^{\infty}|\phi(x,y,z)|^\gamma\ dx \le\ & \left[\int_{-\infty}^{\infty} |\phi(x,y,z)|^{(1-\frac{1}{\lambda})\gamma}|\partial_x\phi(x,y,z)|\ dx\right]^{\frac{1}{2}}\\ &\times\left[\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} |\phi(x,y,z)|^{(1-\frac{1}{\lambda})\gamma}|\partial_y\phi(x,y,z)|\ dxdy\right]^{\frac{1}{2}}\\ &\times\left[\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} |\phi(x,y,z)|^{(1-\frac{1}{\mu})\gamma}|\partial_z\phi(x,y,z)|\ dxdz\right]^{\frac{1}{2}}.\end{aligned}$$

Further integration with respect to y and z yields,

$$\begin{aligned}\int_{\mathbf{R}^3}|\phi(x,y,z)|^\gamma\ dxdydz \le\ & \left[\int_{\mathbf{R}^3} |\phi(x,y,z)|^{(1-\frac{1}{\lambda})\gamma}|\partial_x\phi(x,y,z)|\ dxdydz\right]^{\frac{1}{2}}\\ &\times\left[\int_{\mathbf{R}^3} |\phi(x,y,z)|^{(1-\frac{1}{\lambda})\gamma}|\partial_y\phi(x,y,z)|\ dxdydz\right]^{\frac{1}{2}}\\ &\times\left[\int_{\mathbf{R}^3} |\phi(x,y,z)|^{(1-\frac{1}{\mu})\gamma}|\partial_z\phi(x,y,z)|\ dxdydz\right]^{\frac{1}{2}}.\end{aligned}$$

If Hölder's inequality is applied again, we have

$$\|\phi\|_\gamma^\gamma \le C\,\|\phi\|_\gamma^{(1-\frac{1}{\lambda})\,\frac{\gamma}{2}}\ \|\partial_x\phi\|_\lambda^{\frac{1}{2}}\ \|\phi\|_\gamma^{(1-\frac{1}{\lambda})\,\frac{\gamma}{2}}\ \|\partial_y\phi\|_\lambda^{\frac{1}{2}}\ \|\phi\|_\gamma^{(1-\frac{1}{\mu})\,\frac{\gamma}{2}}\ \|\partial_z\phi\|_\mu^{\frac{1}{2}},$$

which leads to (A.1). □

Lemma A.2. *Let $2 \le q \le 6$ and assume $\phi \in H^1(\mathbf{R}^3)$. Then, there exists a constant $C = C(q)$ such that*

$$\|\phi\|_q \le C\,\|\phi\|_2^{\frac{6-q}{2q}}\ \|\partial_x\phi\|_2^{\frac{q-2}{2q}}\ \|\partial_y\phi\|_2^{\frac{q-2}{2q}}\ \|\partial_z\phi\|_2^{\frac{q-2}{2q}}. \tag{A.6}$$

Proof. This inequality can be obtained by interpolating the trivial inequality $\|\phi\|_2 \le \|\phi\|_2$ and (A.2) with $\mu = 2$, namely

$$\|\phi\|_6 \le C\, \|\phi_x\|_2^{\frac{1}{3}}\, \|\phi_y\|_2^{\frac{1}{3}}\, \|\phi_z\|_2^{\frac{1}{3}}.$$

□

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